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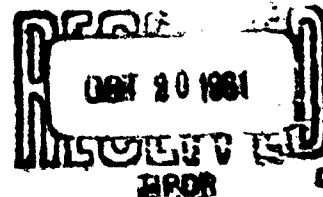
The **RAND** *Corporation*
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U. S. AIR FORCE

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RESEARCH MEMORANDUM

OPTIMAL TIMING IN MISSILE LAUNCHING:
A GAME-THEORETIC ANALYSIS

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SUMMARY

Suppose that Blue plans to launch a missile during some specified time interval. Prior to a launching, Blue must expose the missile for a time period, during which the missile is vulnerable to attack and may be destroyed by Red. Prior to its exposure, Blue's missile is assumed to be underground, where it may be pinned down for a time interval or possibly destroyed by a Red missile. What is the optimal time for Blue to launch his missile, and when should Red attack Blue?

This scheduling problem is formulated and analyzed as a two-person game. We consider various game models in which the above three types of vulnerability are introduced as stochastic elements. The optimal strategies, i.e., the optimal launching times, are described.

CONTENTS

OPTIMAL TIMING IN MISSILE LAUNCHING:
A GAME-THEORETIC ANALYSIS

1. INTRODUCTION

The problem of scheduling the launching of a single missile that must first be exposed for a finite time interval, during which it may be attacked by an enemy missile, has been formulated and solved as an infinite game in [1]. This scheduling problem was subsequently generalized to an arbitrary number of Blue and Red missiles, and the generalized problem was solved in [2] by analyzing an equivalent finite game. In each case it was assumed that Blue was vulnerable to attack only during the exposure period.

We now wish to study the scheduling problem on the assumption that Blue is vulnerable to attack before exposure — i.e., while Blue is underground — as well as during exposure. Additionally, we wish to study the effect of pindown on Blue's optimal scheduling. Finally, we wish to analyze the effect of the reliability of Red's missiles on Blue's schedule.

Accordingly, in this paper we shall first describe a series of game models to study the above factors. We shall then formulate these models mathematically, and shall present the solution to some of them.

2. THE GAME SITUATION

The general game situation that we wish to model and analyze is the following. Blue has B missiles that he wishes to launch before T

minutes have expired, and Red has R missiles with which to attack Blue's missiles during these T minutes. In order to launch a missile, Blue must expose it for L minutes, during which time it may be destroyed (killed while soft) by a Red missile. Prior to exposure, while Blue's missile is underground, the missile may be destroyed (killed while hard), or the missile launching may be postponed for a time interval (pinned down) if a Red missile arrives during this hardened period. Further, a given Red missile may not be reliable and thus may fail to function.

Now in this game situation Blue's object is simply to launch his missiles. Hence, we shall make the payoff to Blue the number of missiles Blue succeeds in launching.

It is clear that a strategy for Blue is a choice of B points on the time interval [0, T] at which to begin the exposure of his missiles. Similarly, a strategy for Red is a choice of R points on the same time interval when Red's missiles are to arrive at Blue's sites. Depending on the amount of information we permit the two players, these choices of times may be functions of the number of missiles Blue and Red have.

Since a strategy is a choice of points on an interval, the game is an infinite game. Generally, however, we may consider an equivalent finite game in which Red chooses R intervals and Blue chooses B intervals. In most cases it will be easier to solve the equivalent finite game than the original infinite game.

3. DEFINITIONS OF PARAMETERS

In the above game it is assumed that both sides know values of the following parameters:

T = number of minutes Blue and Red have to launch their missiles

   = duration of game.

L = number of minutes a Blue missile must be exposed prior to launching = length of exposure period.

N = $[T/L]$ = greatest integer less than or equal to T/L

   = number of exposure periods available in game.

B = number of missiles Blue has at start of game.

R = number of missiles Red has at start of game.

r = probability that a given Red missile is reliable.

A = probability that Blue's missile is destroyed by a Red missile arriving during Blue's missile exposure period.

U = probability that Blue's missile is destroyed by a Red missile arriving when Blue's missile is underground.

F = probability that fallout is produced by a Red missile arriving when Blue's missile is underground; the effect of fallout is to pin down Blue's missile underground for f subsequent periods.

U' = $1 - (1 - U)(1 - F)$ = probability that Blue's missile is either destroyed underground or pinned down by a Red missile arriving when the Blue missile is underground.

$\alpha = 1 - rA$ = probability that Blue's missile survives if exposed when a Red missile is scheduled to arrive.

$\beta = 1 - rU$ = probability that Blue's missile survives if underground when a Red missile is scheduled to arrive.

$\gamma = 1 - rU' = 1 - r[1 - (1 - U)(1 - F)]$ = probability that Blue's missile survives, and is not pinned down, if underground when a Red missile is scheduled to arrive.

4. MODEL I. BLUE VULNERABLE PRIOR TO AND DURING EXPOSURE

In [1] and [2] it was assumed that Blue's missiles are vulnerable to attack only during the exposure period. We shall now assume that the missiles are also vulnerable to attack during the period prior to exposure, i.e., when the missiles are underground or "hard." Further, we shall treat the attack outcomes as random variables, but shall omit the pindown effect in this model.

Suppose Blue and Red have one missile each, i.e., $R = B = 1$. Then a strategy for Blue in the equivalent finite game is a choice of a time period i, where $1 \leq i \leq N$, for Blue to expose his missile for L minutes. A strategy for Red is a choice of a time period j, where $1 \leq j \leq N$, during which Red's missile is to arrive at Blue's launching site. Since the payoff is +1 if Blue launches his missile and 0 otherwise,

the payoff to Blue corresponding to each pair of strategies is given by the matrix

$$
M = (a_{ij}) = \begin{pmatrix}
\alpha & 1 & 1 & \cdot & \cdot & \cdot & 1 \\
\beta & \alpha & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\beta & \beta & \beta & \cdot & \cdot & \alpha & 1 \\
\beta & \beta & \beta & \cdot & \cdot & \beta & \alpha
\end{pmatrix}
$$

Each row of this matrix represents one of the N possible exposure periods for Blue. Each column represents one of the N possible arrival periods for Red.

In order to solve this game, we transform the payoff matrix M to M' by setting

$$
a'_{ij} = \frac{a_{ij} - \alpha}{1 - \alpha} \ ,
$$

and defining

$$
\rho = \frac{\beta - \alpha}{1 - \alpha} \ .
$$

Then the transformed payoff matrix becomes

$$M' = (a'_{ij}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ \rho & 0 & 1 & \cdots & 1 \\ \rho & \rho & 0 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho & \rho & \rho & \cdots & 1 \\ \rho & \rho & \rho & \cdots & 0 \end{pmatrix}$$

By solving M' we obtain the following results for the original game:

(i) The optimal strategy for Blue is

$$x_i = \frac{\rho^N (1 - \rho)}{(1 - \rho^N)} \frac{1}{\rho^i}, \quad \text{for } 1 \le i \le N.$$

(ii) The optimal strategy for Red is

$$y_j = \frac{1 - \rho}{\rho(1 - \rho^N)} \rho^j, \quad \text{for } 1 \le j \le N.$$

(iii) The value of the game is given by

$$v = \frac{\beta - \rho^N}{1 - \rho^N} = 1 - \frac{rU}{1 - (1 - \frac{U}{A})^N}.$$

For optimal scheduling, each player randomizes over each of the N periods. Note, however, that the probabilities associated with each period are different for the two players. As can be seen above, Blue's probability of launching increases geometrically with time, and Red's probability of arrival decreases geometrically with time.

5. MODEL II. BLUE VULNERABLE PRIOR TO

AND DURING EXPOSURE AND PINDOWN

Let us now assume that Blue's missile is subject to pindown as well
as being vulnerable to attack prior to and during exposure. Suppose Blue
and Red have one missile each, i.e., $R = B = 1$. Then a strategy for
Blue is a choice of time x to begin exposing his missile, where
$0 \leq x \leq T - L$. A strategy for Red is a choice of a time y, where
$0 \leq y \leq T$, for Red's missile to arrive at Blue's launching site. If Blue
is attacked when his missile is underground, then the probability that Blue's
missile will be destroyed is U, and the probability that his missile will be
pinned down for f subsequent periods is F. Let us assume that $f = 2$.

From the above description of the game, it is clear that each player
has an infinite number of strategies. The payoff to Blue, $M(x, y)$, for
each pair of strategies, (x, y), is as follows:

$$M(x, y) = \begin{cases} 1 & \text{if } x + L \leq y \leq T, \\ \alpha & \text{if } x \leq y < x + L, \\ \beta & \text{if } x \geq y, \quad y \leq T - 3L, \\ \gamma & \text{if } x \geq y \geq T - 3L. \end{cases}$$

Although this is an infinite game, it turns out that only a finite
number of strategies will be involved in an optimal mixed strategy for
each player. In order to describe the optimal mixed strategies, we set,
as before,

$$\rho = \frac{\beta - \alpha}{1 - \alpha} .$$

Now let us define

$$\lambda = \frac{\gamma - \alpha}{1 - \alpha} ,$$

$$\mu = \lambda + \lambda^2 + \rho\lambda^2 ,$$

and

$$K = \mu \frac{(1 - \rho^{N-3})}{1 - \rho} + \frac{1 - \lambda^3}{1 - \lambda} .$$

It can be verified that the value of the game is given by

$$v = 1 - \frac{r \, AU}{A(1 - \rho^{N-3}) + U\rho^{N-3}\left(\frac{1 - \lambda^3}{1 - \lambda}\right)} .$$

An optimal mixed strategy for Blue consists of choosing one of the following N points in time at random:

$$0, \; L, \; 2L, \; \ldots, \; (N-4)L, \; (N-3)L, \; (N-2)L, \; (N-1)L.$$

These points have the following probabilities, respectively:

$$\frac{\mu}{K}\rho^{N-4}, \; \frac{\mu}{K}\rho^{N-5}, \; \frac{\mu}{K}\rho^{N-6}, \; \ldots, \; \frac{\mu}{K}, \; \frac{\lambda^2}{K}, \; \frac{\lambda}{K}, \; \frac{1}{K} .$$

An optimal mixed strategy for Red consists of choosing one of the following N points in time at random:

$$L, \; 2L, \; 3L, \; \ldots, \; (N-3)L, \; (N-2)L, \; (N-1)L, \; NL.$$

These N points have the following probabilities, respectively:

$$\frac{1}{C}, \quad \frac{\rho}{C}, \quad \frac{\rho^2}{C}, \quad \ldots, \quad \frac{\rho^{N-4}}{C}, \quad \frac{\rho^{N-3}}{C}, \quad \frac{\lambda\rho^{N-3}}{C}, \quad \frac{\lambda^2\rho^{N-3}}{C},$$

where

$$C = \frac{1 - \rho^{N-3}}{1 - \rho} + \frac{1 - \lambda^3}{1 - \lambda} \rho^{N-3}.$$

As in the previous model, Blue's probability of launching increases with time, and Red's probability of attack decreases with time.

6. MODEL III. EQUIVALENT FINITE GAME

The preceding infinite game is equivalent to the following finite game: Blue chooses one of the N time intervals to expose his missile for L minutes. Red chooses one of the N time intervals during which Red's missile is to arrive at Blue's launching site. Thus each player has N strategies. Let the pindown, if it occurs, be for f periods. Then the N x N payoff matrix becomes

$$M = (a_{ij}) = \begin{pmatrix} \alpha & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \beta & \alpha & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta & \beta & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \alpha & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \gamma & \alpha & 1 \\ \beta & \beta & \beta & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \gamma & \gamma & \alpha \end{pmatrix}.$$

In order to solve this game, it is convenient to make the following transformation:

$$a'_{ij} = \frac{a_{ij} - \alpha}{1 - \alpha} \; .$$

The payoff matrix for the transformed game now becomes

$$M' = (a'_{ij}) = \begin{pmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \rho & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \rho & \rho & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \lambda & 0 & \cdot & 0 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho & \rho & \rho & \cdot & \cdot & \cdot & \cdot & \lambda & \cdot & \cdot & \lambda & 0 & 1 \\ \rho & \rho & \rho & \cdot & \cdot & \cdot & \cdot & \lambda & \cdot & \cdot & \lambda & \lambda & 0 \end{pmatrix} .$$

Let $Y = (y_1, y_2, \ldots, y_N)$ be an optimal mixed strategy for Red, and let v' be the value of the M' game. We have the following relations:

$$\rho \sum_{j=1}^{i-1} y_j + \sum_{j=i+1}^{N} y_j = v' \qquad \text{if } i \le N - f - 1,$$

$$\rho \sum_{1}^{N-f-1} y_j + \lambda \sum_{N-f}^{i-1} y_j + \sum_{i+1}^{N} y_j = v' \qquad \text{if } i \ge N - f.$$

Solving these equations for y_j, we get

$$y_j = \begin{cases} \rho^{j-1}(1 - v') & j \le N - f, \\ \lambda^{j-N+f} \rho^{N-f-1}(1 - v') & j \ge N - f + 1, \end{cases}$$

where v' satisfies the following relationship:

$$(1 - v') \left[\frac{1 - \rho^{N-f}}{1 - \rho} + \lambda \rho^{N-f-1} \frac{(1 - \lambda^f)}{1 - \lambda} \right] = 1.$$

Let $X = (x_1, x_2, \ldots, x_N)$ be an optimal mixed strategy for Blue. It follows that

$$\sum_1^{j-1} x_i + \rho \sum_{j+1}^{N} x_i = v' \qquad \text{if } j \leq N - f - 1,$$

$$\sum_1^{j-1} x_i + \lambda \sum_{j+1}^{N} x_i = v' \qquad \text{if } j \geq N - f.$$

From the above equations, we get Blue's optimal strategy:

$$x_i = \begin{cases} \dfrac{1}{\rho^{i-1}} \left(1 - \dfrac{v'}{\rho} \right) & \text{if } i \leq N - f, \\[4mm] \dfrac{1}{\lambda^{i-N+f}} \dfrac{1}{\rho^{N-f-1}} \left(1 - \dfrac{v'}{\rho} \right) & \text{if } i \geq N - f + 1. \end{cases}$$

In order to determine the value v of the original game, we make use of the fact that

$$v = (1 - \alpha)v' + \alpha,$$

and of the relationship (1) for v'. We get

$$v = 1 - \frac{1 - \alpha}{\dfrac{1 - \rho^{N-f-1}}{1 - \rho} + \rho^{N-f-1} \dfrac{(1 - \lambda^{f+1})}{1 - \lambda}} .$$

Perhaps the most outstanding feature of the optimal strategies is that Blue and Red randomize over their respective strategies in opposite

ways. Blue's probability of missile exposure increases with time; i.e., he tends to postpone action. Red's probability of missile firing decreases with time; i.e., he tends to fire early.

7. MODEL IV. RED HAS R RELIABLE MISSILES

Suppose Blue has one missile and Red has several missiles, but each missile is reliable. Since we have assumed that Blue knows the number of missiles that Red has at the beginning of the game, it is reasonable to assume that Blue knows at any given time the number of missiles Red has. Further, Red may play at each instant as if Blue has one missile. This yields a recursive game.

Let $v(m, n)$ define the value of the game in which Red has n missiles at the beginning of the m—period game. Then at the initial period of this game Red has $n+1$ choices: n, $n-1$, ..., 2, 1, 0 missiles to arrive in that period. Blue has two choices: 0 or 1 missile to expose in that period. The payoff is given by the following $2 \times (n+1)$ matrix:

Blue	Red Choice				
Choice	n	$n-1$	$\cdots$	1	0
0	α^n	α^{n-1}	$\cdots$	α	1
1	β^n	$\beta^{n-1}v(m-1, 1)$	$\cdots$	$\beta v(m-1, n-1)$	$v(m-1, n)$

We have the initial condition $v(m, 0) = 1$. Using the above matrix, we can compute values of $v(m, n)$ recursively for all $m \leq N$ and $n \leq R$.

We can also make use of the result from the previous section that

$$v(m, 1) = 1 - \frac{1 - \beta}{1 - \left(\frac{\beta - \alpha}{1 - \alpha}\right)^m} = \frac{\beta - \rho^n}{1 - \rho^n} .$$

It is obvious that

$$v(1, n) = \alpha^n.$$

For example, in order to compute $v(2, 2)$, we use the payoff matrix

$$\begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \alpha\beta & \alpha^2 \end{pmatrix} .$$

For the special case in which $A = 1$, i.e., an exposed missile will be destroyed when a Red missile arrives, we can obtain an explicit solution. First, it is evident that Red should never salvo. Hence a strategy for Red is the choice of zero or one missile during each period. This yields a series of 2×2 games that can be solved recursively.

We can also solve this special case, $A = 1$, directly. A strategy for Blue is a choice of a time period to expose this missile. A strategy for Red is a choice of R of the N time periods for his R missiles to arrive. For example, if $R = 2$ and $N = 4$, we have the following strategies and payoffs:

Blue	Red Strategy					
Strategy	43	42	41	32	31	21
4	0	0	0	1	1	1
3	0	β	β	0	0	1
2	β^2	0	β	0	β	0
1	β^2	β^2	0	β^2	0	0

An optimal strategy for Blue is to expose his missile at period 4, 3, 2, and 1 with relative frequencies

$$\beta^3, \quad \beta^2, \quad \beta, \quad 1,$$

respectively.

An optimal strategy for Red is to choose one of his six strategies with relative frequencies

$$1, \quad \beta, \quad \beta^2, \quad \beta^2, \quad \beta^3, \quad \beta^4.$$

In particular, if we define y_4 to be the probability that Red's first missile arrives during period 4, then

$$y_4 = \frac{1 - \beta^2}{1 - \beta^4}.$$

The value of the game is given by

$$v = \frac{\beta^2 - \beta^4}{1 - \beta^4}.$$

By the above procedure, the value is readily shown for arbitrary R and N, where $R \leq N$, to be

$$v = \frac{\beta^R - \beta^N}{1 - \beta^N} = \frac{\beta^R - \rho^N}{1 - \rho^N} .$$

The optimal relative frequencies for Blue are

$$\beta^{N-1}, \ \beta^{N-2}, \ \ldots, \ \beta, \ 1.$$

An optimal strategy for Red, if expressed explicitly, is very complicated. At any stage (R, N), however, the optimal strategy has the property that

$$y_N = \frac{1 - \beta^R}{1 - \beta^N} .$$

As $\beta \rightarrow 1$, we obtain the uniform distribution given in [2].

8. MODEL V. RED'S MISSILES ARE NOT RELIABLE

We shall now generalize the preceding model by assuming that $r < 1$, i.e., that Red's missiles may not be reliable, and $A < 1$. This changes the complete-information game to a game with incomplete information. Nevertheless, some of the results of the preceding game carry over. We shall describe some partial results regarding this type of game.

Suppose that $R = 2$ and $N \geq 3$. Let B_0 be the Blue mixed strategy defined by the relative frequencies $\rho^{N-1}, \ \rho^{N-2}, \ \ldots, \ \rho^2, \ \rho, \ 1$. It can be shown that if Blue plays B_0, then Red should not salvo. Thus, if

$N = 4$ and if we omit the Red salvo strategies, we have the following payoff matrix:

$$
\begin{array}{c c c c c c c}
 & 43 & 42 & 41 & 32 & 31 & 31 \\
4 & \alpha & \alpha & \alpha & 1 & 1 & 1 \\
3 & \alpha\beta & \beta & \beta & \alpha & \alpha & 1 \\
2 & \beta^2 & \alpha\beta & \beta & \alpha\beta & \beta & \alpha \\
1 & \beta^2 & \beta^2 & \alpha\beta & \beta^2 & \alpha\beta & \alpha\beta
\end{array}
$$

Each of these six Red strategies yields the same expected payoff against Blue's B_0 strategy. Further, if

$$\alpha \leq \frac{\beta^R - \rho^N}{1 - \rho^N} ,$$

where R is arbitrary, it is possible to find a mixture of nonsalvo Red strategies that yields this same expected payoff against each of Blue's pure strategies. Moreover, the nonsalvo strategies dominate the salvo strategies if Blue uses his optimal strategy B_0.

Thus, for arbitrary R and N, the value of the game is

$$v = \frac{\beta^R - \rho^N}{1 - \rho^N} .$$

Hence Blue's optimal strategy is independent of Red's strength.

In this game Blue would get information at the end of each period only if a reliable Red missile had been fired during that period. Blue would therefore know which subgame he is playing. If Red's missile

had been unreliable, however, then Blue would not know Red's resources for the remaining periods.

Suppose Blue has B missiles, B > 1, which may be launched only one per time period. Because of the information about an opponent's resources, Blue and Red have a very large number of strategies. If, however,

$$\alpha \le \frac{\beta^R - \rho^N}{1 - \rho^N} \quad \text{and} \quad B \le \frac{1 - \rho^N}{1 - \rho},$$

then we can show that, for arbitrary N and R, the optimal scheduling for each side is independent of the opponent's resources.

9. MODEL VI. FINITE GAME WITH PINDOWN (B = 1)

In the case of the finite game with pindown a possibility, both sides have incomplete information, but we shall assume that neither side can act on the information during the play of the game. Each side formulates a firing schedule in advance and will not change it during the game, except to postpone exposure when forced to do so by pindown.

Such a game has a large number of strategies. For example, if N = 4, R = 2, we have the following payoff matrix:

Blue Strategy	Red Strategy									
	44	43	42	41	33	32	31	22	21	11
4	α^2	α	α	α	1	1	1	1	1	1
3	β^2	$\gamma\alpha+(\beta-\gamma)\gamma$	$\gamma\cdot 1+(\beta-\gamma)\gamma$	$\gamma\cdot 1+(\beta-\gamma)\alpha$	α^2	α	α	1	1	1
2	β^2	$\beta\gamma$	$\alpha\gamma+(\beta-\gamma)\gamma$	$\gamma\cdot 1+(\beta-\gamma)\alpha$	γ^2	$\gamma\alpha$	γ	α^2	α	1
1	β^2	$\beta\gamma$	$\beta\gamma$	$\beta\alpha$	γ^2	γ^2	$\gamma\alpha$	γ^2	$\gamma\alpha$	α^2

We note from the payoff matrix that if Blue chooses his 3 strategy and Red chooses his 43 strategy, then the payoff is $\gamma\alpha + (\beta - \gamma)\gamma$. This is obtained as follows: Either there is pindown or there is none, due to Red's first missile arriving in the 4th period. If there is no pindown, Blue's missile survives while underground with probability γ. Then he will proceed as scheduled to launch his missile during the 3rd period. With probability α he will survive Red's second missile scheduled to arrive in that period.

On the other hand, if there is pindown from Red's first missile, then Blue must postpone his launching to period one, the last period. In that case, the probability that he will survive Red's second missile is γ.

In this way, each Red missile arriving before the $(f+1)$th period from the end gives rise to 2 cases (pindown or no pindown). The entries in the payoff matrix for R missiles can be expressed as weighted sums of two entries taken from game matrices for smaller numbers of Red missiles.

While these large games look hopelessly complicated at first glance, there is hope of solution for certain ranges of the probabilities when R is small. Approximations to this model can be worked out sequentially.

As an approximation to this model, we can set up the corresponding sequential game in which both sides have complete information at each stage. At each stage or period, Red will have $R' + 1$ choices, and Blue will have two choices at most. Recursive relations can be built up from matrices similar to the following game matrix illustrating the case R = 3, N = 4:

Blue Strategy	Red Strategy			
	(4, 3)	(4, 2)	(4, 1)	(4, 0)
(4, 1)	α^3	α^2	α	1
(4, 0)	$\gamma^3 V(3, 0) +$ $(\beta - \gamma)^3 V(1, 0)$	$\gamma^2 V(3, 1) +$ $(\beta - \gamma)^2 V(1, 1)$	$\gamma V(3, 2) +$ $(\beta - \gamma) V(1, 2)$	$V(3, 3)$

where $V(N', R')$ = the value of the subgame with N' periods and R' Red missiles left.

Given the parameters, we can solve the games by using these recursive relations. Figures 1 and 2 present the game values for two different sets of parameters.
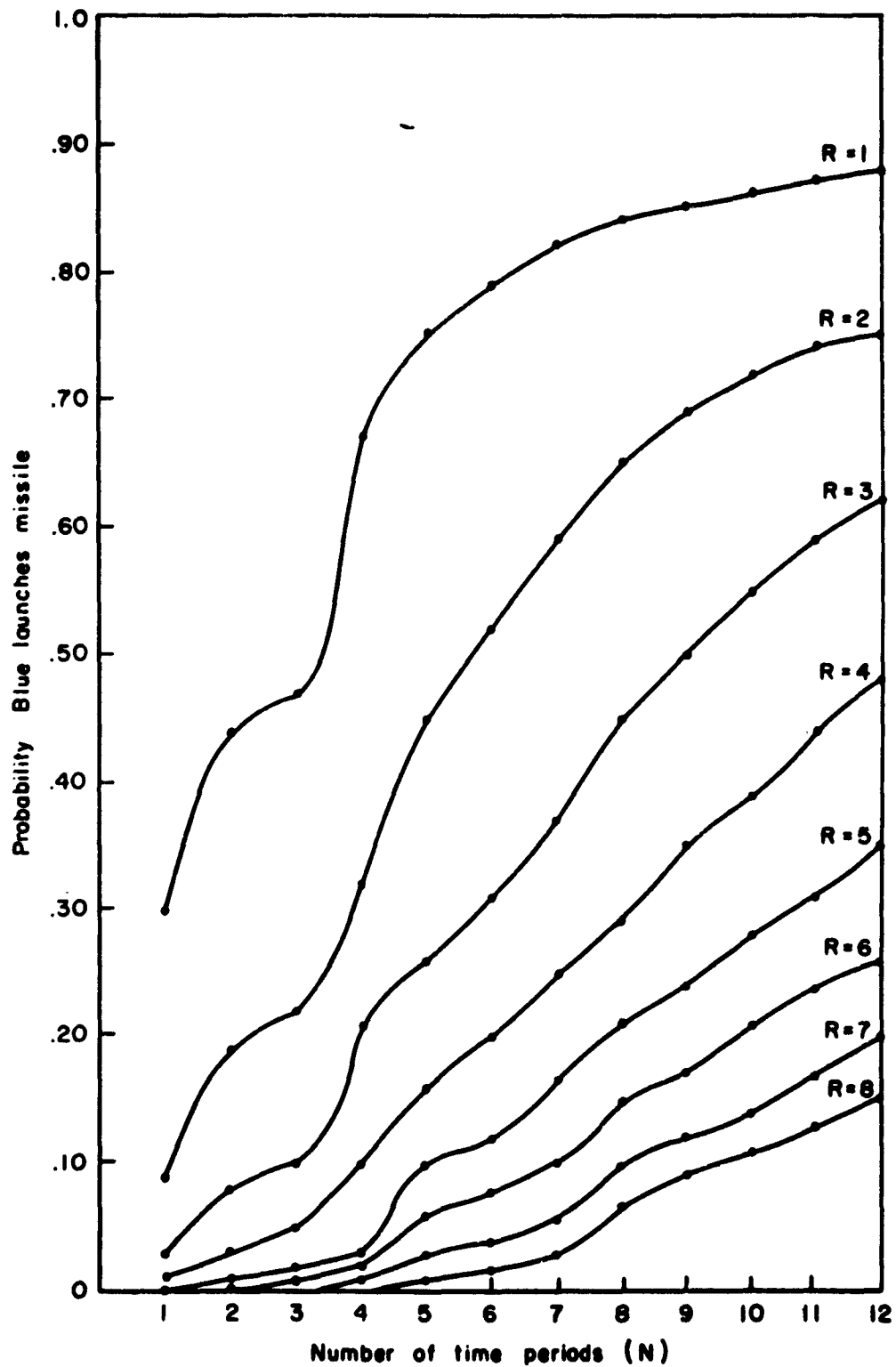
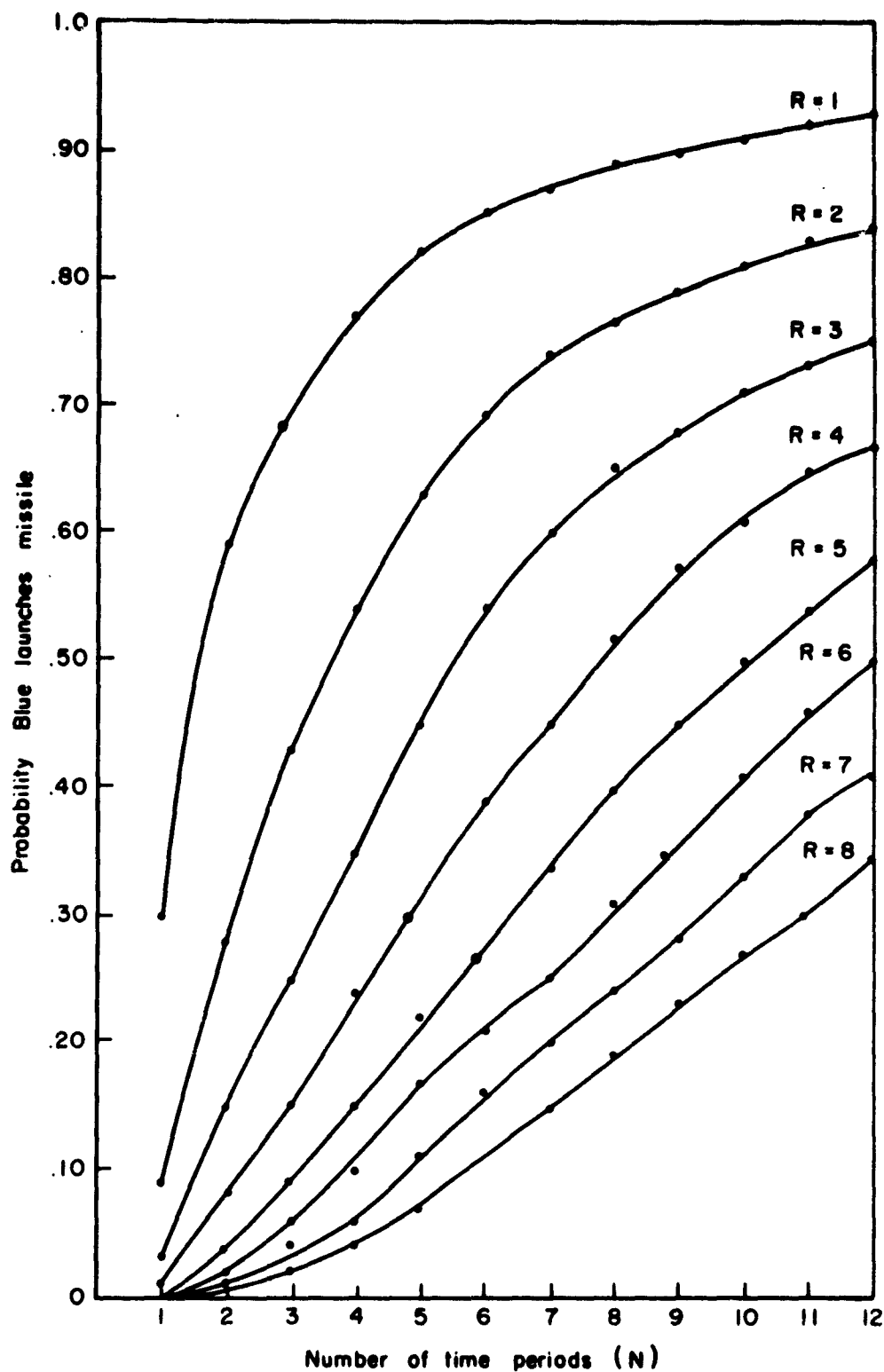Fig. I — Game values for parameters
r = 0.7, U = 0.15, F = 0.7, A = I

Fig. 2— Game values for parameters
r = 0.7, U = 0.04, F = 0.27, A = I

REFERENCES

1. Dresher, Melvin, Optimal Tactics in Delayed Firing: A Game—
   theoretic Investigation (U), The RAND Corporation, Research
   Memorandum RM—1886 (ASTIA No. AD 150656), March 28, 1957
   (Confidential).

2. Johnson, S. M., A Game Solution to a Missile—launching Scheduling
   Problem, The RAND Corporation, Research Memorandum
   RM—2398 (ASTIA No. AD 237785), November 25, 1959.

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is assumed to be underground, where it may be planned down for a time
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determine the optimal time for Blue to launch his missile and for Red to
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See also RM-1866 and RM-2396

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